External noise in periodically forced Rayleigh-Bénard convection

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The effect of internal and external noise sources on the onset of Rayleigh-Bénard convection is studied for the case of time-periodic forcing. Internal noise is modeled through the inclusion of an additive white Gaussian noise field in the Swift-Hohenberg equation. External noise is modeled as multiplicative Gaussian noise. A comparison with available theoretical and experimental data is presented. We conclude that noise in the control parameter of the Swift-Hohenberg equation is not likely to explain the difference between the thermal noise intensity and the one needed to make theory fit the experiments. [S1063-651X(96)04512-6]

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Very recently, García-Ojalvo *et al.* [1] have studied the Swift-Hohenberg equation in the presence of multiplicative noise. They aim to describe a situation in which a noise has been superimposed on the temperature gradient between the two plates of a Rayleigh-Bénard cell.

The influence of noise on the onset of Rayleigh-Bénard convection has been studied over the years [2,3] both experimentally [4–6] and theoretically [2,7]. However, rigorous analytic results are scarce due to the technical difficulties of the underlying equations, and numerical simulations are often very expensive in CPU time, so the problem still poses several unanswered questions.

One of these concerns is the noise strength. For fluids in a Rayleigh-Bénard cell, the thermal noise strength can be shown to be $F_{\rm th}\sim k_B T/\rho d\nu^2$ for both free-slip [7] and rigid [8] boundary conditions, where ρ is the mass density, d the plate separation, and ν the kinematic viscosity. Below the convective onset, Wu *et al.* [6] have found that the noise power needed to explain their measurements agrees with this prediction. But in experiments that modulate the control parameter through the onset [5], a much greater (by a factor $\sim 10^4$) noise strength is needed, and there is no explanation yet for this discrepancy. Despite a great deal of work by many authors, a mechanism enabling thermal noise alone to provide the driving force for the onset of convection in these *dynamic* experiments, has not still been proposed [8].

A possible way out of this conundrum is the inclusion of external (nonthermal) noise sources. In Ref. [1] García-Ojalvo *et al.* addressed this issue for the static case. In this work we study the effect of external noise on the Rayleigh-Bénard convection with sinusoidal modulation of the control parameter, which has been studied experimentally [5] and theoretically [9] in the context of internal noise. We introduce a self-consistent approximation to the corresponding Swift-Hohenberg equation, which allows an efficient numerical calculation of the self-correlation of the order parameter. We show that our approach reproduces the known results both for the dynamic case without external noise [9,10] and for the static case with external noise [1]. Then we estimate the strength of the external noise that would be needed to fit the experimental results of Meyer *et al.* [5], and compare it with the intensity of the fluctuations in the control parameter compatible with recent experiments [6].

We begin by briefly recalling the standard theory of Rayleigh-Bénard convection. In the typical experimental setting, we have a fluid between two horizontal plates, forming a cell of lateral size $\widetilde{L}$, height d, and aspect ratio $L=\widetilde{L}/d$. The density of the fluid is ρ, its temperature is T, its velocity is $\mathbf{u}$, and its pressure is p. The kinematic viscosity is denoted by ν, the thermal diffusivity by κ, and the gravitational acceleration by $\widetilde{g}$.

The description of the system proceeds by writing the Navier-Stokes equations in the Boussinesq approximation [3], with *rigid* boundary conditions $\mathbf{u}(z=0)=\mathbf{u}(z=d)=\mathbf{0}$ and $T(z=0)=T(z=d)+\Delta T$, setting $\theta=T-T_0(z)$ (with T_0 the temperature profile in the conductive state), and introducing adimensional variables by the scaling $t\to(\kappa/d^2)t$, $l\to l/d$, $\Delta T\to R$, where $R=(\alpha\widetilde{g}d^3\Delta T)/(\kappa\nu)$ is the Rayleigh number. Defining the Prandtl number $\sigma=\nu/\kappa$, the evolution equations for the temperature and velocity fields then read

$$\sigma^{-1}(\partial_t+\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=-\boldsymbol{\nabla}p+\theta\hat{\mathbf{z}}+\boldsymbol{\nabla}^2\mathbf{u}, \tag{1a}$$

$$(\partial_t+\mathbf{u}\cdot\boldsymbol{\nabla})\theta=Ru_z+\boldsymbol{\nabla}^2\theta, \tag{1b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0, \tag{1c}$$

with boundary conditions $\mathbf{u}=\theta=0$ at $z=0,\ d$.

Near the convection onset, linear stability analysis of these equations gives the critical Rayleigh number $R_c=1707.76$ and the critical wave number $q_0=3.117$ [3].

Introducing the order parameter $\Psi(\mathbf{r},t)$ with

$$\mathbf{u},\theta\propto\Psi\times(\text{vertical eigenfunctions}) \tag{2}$$

[here $\mathbf{r}$ stands for $(\mathbf{x,y})$] and applying $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$ to Eq. (1a) gives the Swift-Hohenberg (SH) equation [2,3]

$$\tau_0\partial_t\Psi(\mathbf{r},t)=[\epsilon-\widetilde{\xi}_0^4(q_0^2+\boldsymbol{\nabla}^2)^2-g\Psi^2(\mathbf{r},t)]\Psi(\mathbf{r},t)+\sqrt{D}\ \xi(\mathbf{r},t), \tag{3}$$

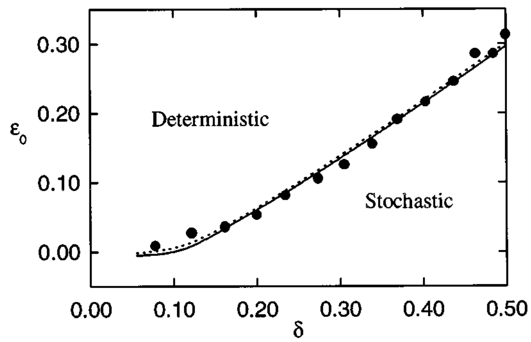

FIG. 1. Order-disorder transition line for the one-mode equation (4). The circles are the experimental data of Meyer *et al.* [5]. The continuous line is the prediction of the MF approximation for $\gamma=5\times10^{-7}$. The dotted line is the result of 10^5 Monte Carlo runs with the same γ; the linewidth is greater than the standard deviation.

where $\tau_0=(\sigma+0.5117)/(19.65\sigma)$, $\xi_0^2=0.148$, $\widetilde{\xi}_0^4=\xi_0^2/(4q_0^2)$, $\overline{g}=0.6995-0.0047\sigma^{-1}+0.0083\sigma^{-2}$, $g=\overline{g}/3$, $\epsilon=(R-R_c)/R_c$. We have modeled thermal noise as usual, by adding a zero-mean, unit variance Gaussian white noise field $\xi(\mathbf{r},t)$, that is, $\langle\xi(\mathbf{r},t)\rangle=0$, $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$. Its intensity, as given by equilibrium thermodynamics [3], is $D_{\rm th}=2\sigma k_BT/(R_c\rho d\nu^2)\sim10^{-11}$. The smallness of this noise intensity has been experimentally verified [5,6] for ϵ (i.e., ΔT) independent of time and below the convection threshold.

A well-known simplified version of (3) is the one-mode equation [3,4,9]

$$\tau_0\dot{A}=[(\epsilon-\epsilon_c)-\overline{g}A^2]A+\sqrt{\frac{2\gamma}{L}}\xi(t), \quad (4)$$

for the slowest mode A of the amplitude equation [4] which describes the system in a finite cylindrical cell of radius L. Here $\epsilon_c=(\xi_0\pi/L)^2$, $\xi(t)$ is a zero-mean, unit variance Gaussian white noise, and $\gamma\equiv D$ [3].

The experiment we consider was performed by Meyer *et al.* [5]. The working fluid was water, with $\rho=1$, $T\sim300$ K, $\kappa=1.47\times10^{-3}$ cm^2/s, and $\sigma=6$. The cell was cylindrical with radius $\widetilde{L}=3.18$ cm, height $d=0.318$ cm, and rigid boundary conditions. The system was forced to sweep repeatedly through its convective onset by setting $\epsilon(t)=\epsilon_0+\delta\cos\omega t$, with $|\delta|>\epsilon_0>0$ and $\omega=1$. The main quantity measured was the Nusselt number $\mathcal{N}\propto(1/V)\int\langle\Psi^2\rangle\,d\mathbf{r}$. As will be shown later, in our approach $\langle\Psi^2\rangle$ is independent of the coordinates, so the spatial averaging can be omitted.

In this experiment an order-disorder transition was observed. This is a sharp transition in the (ϵ_0,δ) plane between "stochastic" behavior (the convective cell pattern is not reproduced for successive cycles) and "deterministic" behavior (the same convective pattern reappears in successive cycles). This transition is depicted in Fig. 1. The transition line can be defined as the curve on which the order-parameter self-correlation after one period of the external forcing, $\langle\Psi(\mathbf{r},t+2\pi/\omega)\Psi(\mathbf{r},t)\rangle$, is equal to $\langle\Psi^2(\mathbf{r},t)\rangle/2$. Several numeric and (approximate) analytic computations (see, e.g., [3,9,10]) show that the transition line predicted from the standard theory can be made to fit the experimental data, only by taking the internal noise strength as an adjustable parameter and setting it to $D\sim10^4D_{\rm th}$. The immediate conclusion is that if the theory is correct there must be additional noise sources besides the pure thermodynamic noise. The current proposals are hydrodynamic mechanisms that can enhance the thermodynamic noise intensity or make it nonwhite, and external noise sources such as fluctuations in the boundary temperature of the cell. In this work we will consider only the effect of the latter.

A proposed source of external noise are fluctuations in the control parameter ϵ (i.e., in ΔT) [1], due to imperfect temperature stabilization in the cell's upper and lower plates. This amounts to rewriting (3) with

$$\epsilon\ \rightarrow\ \epsilon+\sqrt{d}\,\eta(\mathbf{r},t), \quad (5)$$

where $\eta(\mathbf{r},t)$ is again a zero-mean, unit variance Gaussian white noise field, which we take as independent of $\xi(\mathbf{r},t)$. As usual in this context [1], Eq. (3) with multiplicative noise must be interpreted in the Stratonovich sense.

The main difficulty solving (3) or (4) is their nonlinearity. The usual linearization (see, e.g., Ref. [3], and references therein) around the *deterministic* conductive profile (corresponding to $\Psi\equiv0$) is clearly unsatisfactory for the periodic-forcing experiment, though it gives good results for the static case below convective onset [1,6].

The approximation introduced in this work consists in replacing (3) by

$$\tau_0\partial_t\Psi(\mathbf{r},t)=[\epsilon+\sqrt{d}\,\eta(\mathbf{r},t)-\widetilde{\xi}_0^4(q_0^2+\nabla^2)^2 - g\langle\Psi^2(t)\rangle]\Psi(\mathbf{r},t)+\sqrt{D}\xi(\mathbf{r},t), \quad (6)$$

where $\langle\Psi^2(t)\rangle$ is assumed independent of $\mathbf{r}$ by the (statistic) translational invariance of (3), and is self-consistently computed *a posteriori*. This is in the same spirit of the usual mean-field approximation of statistical mechanics, which assumes that the only important configuration near a critical point is the spatially uniform one [11], hence we call it "mean-field" (MF) approximation. Fourier transforming $\mathbf{r}\rightarrow\mathbf{k}$, and using Novikov's theorem [12] or a direct generalization of the known correlation formulas [13], the instantaneous order-parameter structure factor $S_0(k,t)=\langle\Psi^*(k,t)\Psi(k,t)\rangle$ can be shown to obey (for long times) the equation

$$\frac{\partial}{\partial t}S_0(k,t)=\frac{2}{\tau}\left(\epsilon(t)+d-\widetilde{\xi}_0^4(q_0^2-k^2)^2 - \frac{g}{2\pi}\int_0^\infty S_0(v,t)v\ dv\right)S_0(k,t) + \frac{d}{2\pi\tau^2}\int_0^\infty S_0(v,t)v\ dv+\frac{D}{\tau^2}. \quad (7)$$

Similarly the two-times structure factor $S(k,t,t')=\langle\Psi^*(k,t)\Psi(k,t')\rangle$ can be shown to obey

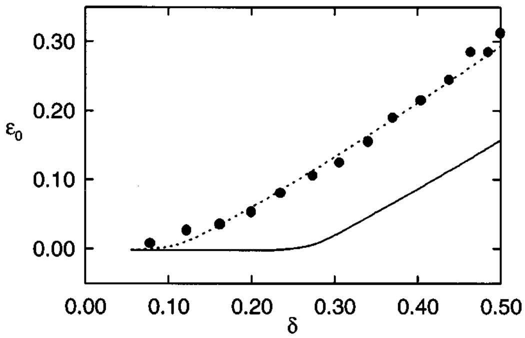

FIG. 2. Order-disorder transition line for the Swift-Hohenberg equation (3). The circles are the experimental data of Meyer *et al.* [5]. The lines are the predictions of the MF approximation for $D=D_{\rm th}=1.172\times10^{-11}$ (continuous line) and $D=5\times10^4 D_{\rm th}$ (dotted line).

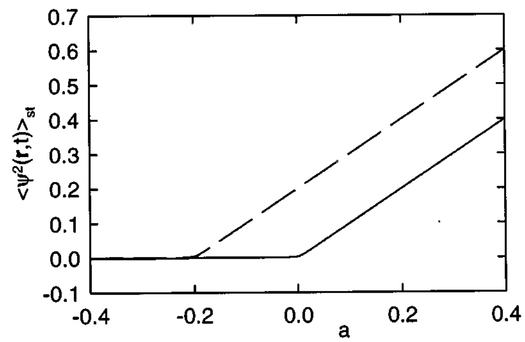

FIG. 4. Steady heat flux versus ϵ for the static case of Ref. [1], for $D=0.001$. The continuous line is the prediction of the MF approximation to the SH equation for $d=0$; the dashed line is the prediction for $d=0.1$.

$$\frac{\partial}{\partial t'}S(k,t,t')=\frac{1}{\tau}\Bigg(\epsilon(t')+d-\tilde{\xi}_0^4(q_0^2-k^2)^2 - \frac{g}{2\pi}\int_0^\infty S_0(v,t')v\,dv\Bigg)S(k,t,t'), \quad (8)$$

with $t'>t$ and initial condition $S(k,t,t)=S_0(k,t)$. Here S and S_0 depend only on $k=|\mathbf{k}|$ because of the (statistic) rotational symmetry of (3). From these quantities we can compute the order-parameter self-correlation

$$\langle\Psi(\mathbf{r},t)\Psi(\mathbf{r}',t')\rangle=\frac{1}{(2\pi)^2}\int e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}S(k,t,t')\,d^2k \quad (9)$$

and its mean-square value

$$\langle\Psi^2(t)\rangle=\frac{1}{2\pi}\int_0^\infty S_0(k,t)k\,dk. \quad (10)$$

The closed character of the evolution equation for S_0 (or $\langle\Psi^2\rangle$) makes the approximation (6) self-consistent. Equations (7) and (8) have time-periodic solutions in the long-time (asymptotic) regime, which provide the self-correlation and mean-square value of the order parameter. In what follows, these integro-differential equations are solved numerically, and the predicted order-disorder transition line is then obtained through Eq. (10).

In order to assess the validity of the MF approximation, we first addressed the one-mode equation (4) with additive noise only, for the parameter values of the experiment of Meyer *et al.* [5]. We computed the order-disorder transition line predicted by the MF approximation to (4), thoroughly similar to that leading from (3) to (6)–(8). Besides, we performed a Monte Carlo integration of Eq. (4) [10,14]. We plotted in Fig. 1 the corresponding transition lines, taking γ as an adjustable parameter as in Refs. [9,10]. The agreement between both approaches is excellent, providing a validation of MF.

Next we computed the transition line predicted from (7)–(10) for the SH equation with additive noise only, and the parameters of Meyer *et al.* [5]. The result is shown in Fig. 2. We observe that fitting the experimental data requires an internal noise intensity $D=5\times10^4 D_{\rm th}$ as usual [3,9,10], but besides this the agreement is as good as for the approaches of these references. This provides further validation of MF.

We then solved (7)–(8) for the SH equation with additive

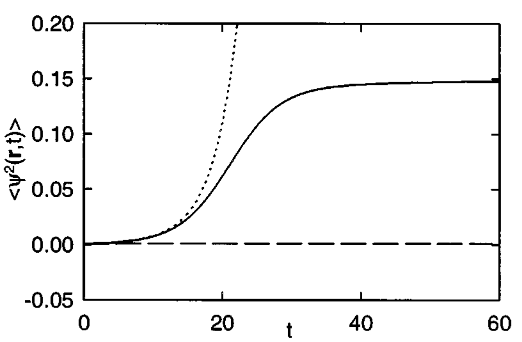

FIG. 3. Transmitted heat flux versus time for the static case of Ref. [1], for $D=0.001$ and $\epsilon=-0.05$. The continuous line is the prediction of the MF approximation to the SH equation for $d=0.1$; the dashed line is the prediction for $d=0$. The prediction for the *linear* SH equation with $d=0.1$ is the dotted line.

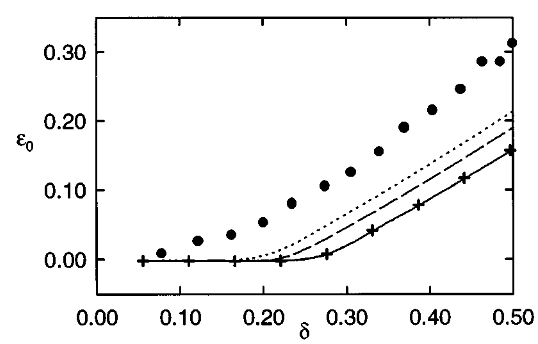

FIG. 5. Order-disorder transition line for the Swift-Hohenberg equation (3). The circles are the experimental data of Meyer *et al.* [5]. The lines are the predictions of the MF approximation for $D=D_{\rm th}=1.172\times10^{-11}$ and successively higher values of $d=0$, $10^6 D_{\rm th}$, $10^8 D_{\rm th}$, and $1.1\times10^8 D_{\rm th}$. The crosses are the MF prediction for $d=10^{-4}$, compatible with the measurements of Ref. [6].

and multiplicative noise, and the parameters of the static case considered in Ref. [1] ($\epsilon = a =$ const, $\tau_0 = \tilde{\xi}_0 = q_0 = g = 1$). In Fig. 3 we plotted the transmitted heat flux [$\propto \langle \Psi^2 \rangle$] as a function of time for internal noise $D = 0.001$ and $\epsilon = -0.05$ (slightly below the convective onset for additive noise only). We see that adding a small external noise $d = 0.1$ anticipates the onset of convection, as in Ref. [1]. We also computed for comparison purposes the flux predicted by the linear approximation to (3) considered in Ref. [1]. In Fig. 4 we plotted the steady heat flux (long-time regime) versus the deterministic control parameter a. The predicted anticipation of the convective threshold for $d > 0$ is obtained.

The agreement with the results of Ref. [1] is excellent. We must note, however, that (except for the linear case in Fig. 3) the results of Ref. [1] have been obtained through a Monte Carlo integration of the full two-dimensional SH equation. It is worth noting that the anticipation of the convective threshold is appreciable only for $d \gtrsim 100D$.

Finally, we computed through (7)–(10) the order-disorder transition line for the parameters of the time-periodic experiment of Meyer *et al.* [5], adding external noise as multiplicative noise in the SH equation, but taking the internal noise to be $D = D_{\rm th}$. The results are plotted in Fig. 5.

We observe that the predicted transition line remains the same as for additive noise up to $d \sim 10^6 D_{\rm th}$, and the experimental transition line would be fitted for $d \sim 10^9 D_{\rm th}$. This situation is far worse than that for purely additive noise, in which the thermal noise intensity must be exaggerated by a factor of $\sim 10^4$ to fit the data.

Besides, it must be noted that $10^9 D_{\rm th} \sim 10^{-2}$, while recent experiments [6] have shown that the lower and upper plate temperatures (thus ϵ) are routinely stabilized better than one part in 10^{-3}. The typical values of ϵ_0 and δ being of order 10^{-1}, so big a value of d is doubtful.

We can conclude therefore that the inclusion of external noise as multiplicative noise in the control parameter of the Swift-Hohenberg equation is not likely to explain the discrepancy between the thermodynamic and observed noise intensities.

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